Number Systems

- **Irrational Numbers**

  - Irrational numbers are those which **cannot** be expressed in the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$.

    Example: $\pi, \sqrt{2}, \sqrt{7}, \sqrt{14}, 0.0202202220\ldots$ are irrational numbers.

    Irrational numbers are the numbers which neither terminate nor repeat.

    Example: $\frac{22}{7}$ **or as 3.14, both of which are rationals.**

- Decimal expansion of a rational number can be of two types:

  (i) Terminating
  (ii) Non-terminating and repetitive

  In order to find decimal expansion of rational numbers we use long division method.

  For example, to find the decimal expansion of $\frac{1237}{25}$.

  We perform the long division of 1237 by 25.

$$
\begin{array}{r|l}
 & \;\;49.48 \\
\hline
25 & 1237.00 \\
 & 100 \\
\hline
 & \;\;237 \\
 & \;\;225 \\
\hline
 & \;\;\;\;120 \\
 & \;\;\;\;100 \\
\hline
 & \;\;\;\;\;\;200 \\
 & \;\;\;\;\;\;200 \\
\hline
 & \;\;\;\;\;\;\;\;0 \\
\hline
\end{array}
$$

  Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal number is terminating.

- **Decimal expansion of irrational numbers**

  - The decimal expansion of an irrational number is non-terminating and non-repeating. Thus, a number whose decimal expansion is non-terminating and non- repeating is irrational.

For example, the decimal expansion of $\sqrt{2}$ is 1.41421…. , which is clearly non-terminating and non-repeating. Thus, $\sqrt{2}$ is an irrational number.

- The number $\sqrt[n]{a}$ is irrational if it is not possible to represent a in the form b^n, where b is a factor of a.

  For example, $\sqrt[6]{12}$ is irrational as 12 cannot be written in the form b^6, where b is a factor of 12.

- **Conversion of decimals into equivalent rational numbers:**

- Non-terminating repeating decimals can be easily converted into their equivalent rational numbers.

  For example, $2.35\overline{961}$ can be converted in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ as follows:
  Let $x = 2.35\overline{961}$
  ∴ $x = 2.35961961...$ … (1)
  On multiplying both sides of equation (1) with 100, we obtain:
  $100x = 235.961961961...$ … (2)
  On multiplying both sides of equation (2) with 1000, we obtain:
  $100000x = 235961.961961961...$ … (3)
  On subtracting equation (2) from equation (3), we obtain:
  $99900x = 235726$
  $\Rightarrow x = \frac{235726}{99900} = \frac{117863}{49950}$
  Thus, $2.35\overline{961} = \frac{117863}{49950}$

- **Irrational numbers between any two rational numbers:**

  There are infinite irrational numbers between any two rational numbers.
  We can find irrational numbers between two rational numbers using the following steps:

  Step 1: Find the decimal representation (up to 2 or 3 places of decimal) of the two given rational numbers. Let those decimal representations be a and b, such that $a < b$.

  Step 2: Choose the required non-terminating and non-repeating decimal numbers (i.e., irrationl numbers) between a and b.

  Example: 0.34560561562563…, 0.3574744744474444… and 0.369874562301…are three irrational numbers beween 0.33 and 0.4.

- **Representation of rational numbers on number line using successive magnification:**

  Example: Visualize $3.\overline{32}$ on the number line, upto 4 decimal places.

Solution:

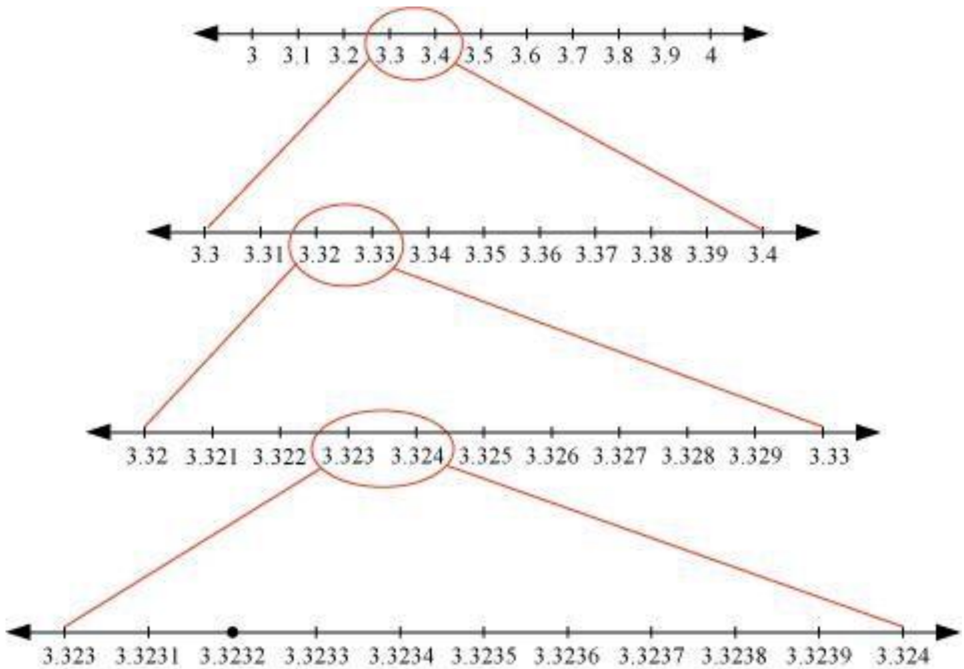
$3.\overline{32} = 3.3232...$
$= 3.3232$ (approximate upto 4 decimal place)

Step 1: As 3 < 3.3232 < 4, so divide the gap between 3 and 4 on the number line into 10 equal parts and magnify the distance between them.

Step 2: As 3.3 **<** 3.3232 < 3.4, so again divide the gap between between 3.3 and 3.4 into 10 equal parts to locate the given number more accurately.

Step 3: As 3.32 < 3.3232 < 3.33 so, we continue the same procedure by dividing the gap between 3.32 and 3.33 into 10 equal parts.

Step 4: Also, 3.323 < 3.3232 < 3.324, so by dividing the gap between 3.323 and 3.324 into 10 equal parts, we can locate 3.3232.



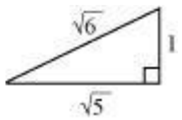
- **Represent irrational numbers on the number line:**

We can represent irrational numbers of the form $\sqrt{n}$ on the number line by first plotting $\sqrt{n-1}$, where *n* is any positive integer.

Example: Locate $\sqrt{6}$ on the number line.

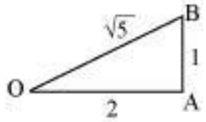
Solution:

As $\sqrt{6} = \sqrt{\left(\sqrt{5}\right)^2 + 1^2}$



To locate $\sqrt{6}$ on the number line, we first need to construct a length of $\sqrt{5}$.

$$\sqrt{5} = \sqrt{2^2 + 1}$$



By Pythagoras theorem, $OB^2 = OA^2 + AB^2 = 2^2 + 1^2 = 5$

$\Rightarrow OB = \sqrt{5}$

Steps:

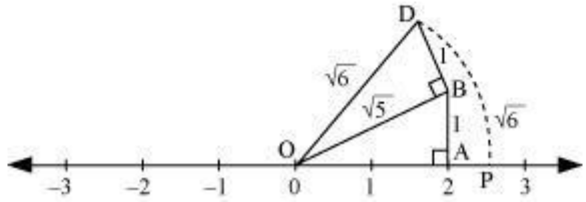
Mark O at 0 and A at 2 on the number line, and then draw AB of unit length perpendicular to OA. Then, by Pythagoras Theorem, $OB = \sqrt{5}$.

Construct BD of unit length perpendicular to OB. Thus, by Pythagoras theorem,

$$OD = \sqrt{\left(\sqrt{5}\right)^2 + 1^2} = \sqrt{6}$$

Using a compass with centre O and radius OD, draw an arc intersecting the number line at point P.

Thus, P corresponds to the number $\sqrt{6}$.



- **Representation of real numbers of the form $\sqrt{n}$ on the number line, where *n* is any positive real number:**

  We cannot represent $\sqrt{n}$ on number line directly, so we will use the geometrical method to represent $\sqrt{n}$ on the number line.

  Example:

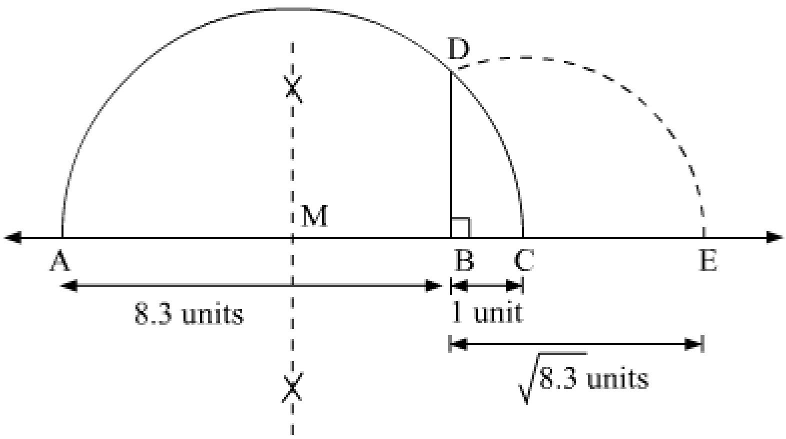
  Represent $\sqrt{8.3}$ on the number line.

  Solution:

  Step 1: Draw a line and mark a point A on it. Mark points B and C such that AB = 8.3 units and BC = 1 unit.

Step 2: Find the mid-point of AC and mark it as M. Taking M as the centre and MA as the radius, draw a semi-circle.

Step 3: From B, draw a perpendicular to AC. Let it meet the semi-circle at D. Taking B as the centre and BD as the radius, draw an arc that intersects the line at E.



Now, the distance BE on this line is $\sqrt{8.3}$ units.

- **Operation on irrational numbers:**

  - **Like terms:** The terms or numbers whose irrational parts are the same are known as like terms. We can add or subtract like irrational numbers only.
  - **Unlike terms:** The terms or numbers whose irrational parts are not the same are known as unlike terms.

We can perform addition, subtraction, multiplication and division involving irrational numbers.

Note:

(1) The sum or difference of a rational and an irrational number is always irrational.
(2) The product or quotient of a non-zero rational number and an irrational number is always irrational.

Example:

(1) $\left(2\sqrt{3}+\sqrt{2}\right)+\left(3\sqrt{3}-5\sqrt{2}\right)$
$=\left(2\sqrt{3}+3\sqrt{3}\right)+\left(\sqrt{2}-5\sqrt{2}\right)$ (Collecting like terms)
$=\left(2+3\right)\sqrt{3}+\left(1-5\right)\sqrt{2}$
$=5\sqrt{3}-4\sqrt{2}$
(2) $\left(5\sqrt{7}-3\sqrt{2}\right)-\left(7\sqrt{7}+3\sqrt{2}\right)$

$= 5\sqrt{7} - 3\sqrt{2} - 7\sqrt{7} - 3\sqrt{2}$

$= 5\sqrt{7} - 7\sqrt{7} - 3\sqrt{2} - 3\sqrt{2}$

$= (5-7)\sqrt{7} - (3+3)\sqrt{2}$ (Collecting like terms)

$= -2\sqrt{7} - 6\sqrt{2}$

(3) $\mathbf{(4\sqrt{5} + 3\sqrt{2}) \times \sqrt{2}}$

$= 4\sqrt{5} \times \sqrt{2} + 3\sqrt{2} \times \sqrt{2}$

$= 4\sqrt{10} + 3 \times 2 \qquad (\sqrt{2} \times \sqrt{2} = 2)$

$= 4\sqrt{10} + 6$

(4) $\mathbf{5\sqrt{6} \div \sqrt{12}}$

$= 5\sqrt{6} \times \frac{1}{\sqrt{12}}$

$= \frac{5 \times \sqrt{2} \times \sqrt{3}}{2 \times \sqrt{3}}$

$= \frac{5}{2}\sqrt{2}$

- **Closure Property of irrational numbers:**

  Irrational numbers are not closed under addition, subtraction, multiplication and division.

  Example: $-\sqrt{2} + \sqrt{2} = 0$, $\sqrt{2} - \sqrt{2} = 0$, $\sqrt{2} \times \sqrt{2} = 2$ and $\frac{\sqrt{2}}{\sqrt{2}} = 1$, which are not an irrational numbers.

- **Identities related to square root of positive real numbers:**
  If *a* and *b* are positive real numbers then

  $\sqrt{ab} = \sqrt{a}\sqrt{b}$

  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

  $(\sqrt{a} + b)(\sqrt{a} - b) = a - b^2$

  $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$

  $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$

  $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$

  $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$

  $(\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d}) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$

  We can use these identities to solve expressions involving irrational numbers.

  Example:

$\left(\sqrt{5}+3\right)\left(\sqrt{5}-3\right)$
$=5-(3)^2$
$=5-9$
$=-4$

- **Rationalization of denominators:**
  - The denominator of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{x}+\sqrt{y}}$ can be rationalized by multiplying both the numerator and the denominator by $\sqrt{x}-\sqrt{y}$, where *a*, *b*, *x* and *y* are integers.
  - The denominator of $\frac{\sqrt{a}+\sqrt{b}}{c+\sqrt{d}}$ can be rationalized by multiplying both the numerator and the denominator by $c-\sqrt{d}$, where *a*, *b*, *c* and *d* are integers.

  Note: $\sqrt{x}-\sqrt{y}$ and $c-\sqrt{d}$ are the conjugates of $\sqrt{x}+\sqrt{y}$ and $c+\sqrt{d}$ respectively.

Example: Rationalize $\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}$

Solution:

$\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}$

$=\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

$=\frac{2\sqrt{2\times5}-2\sqrt{2\times3}}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$ $\quad\left[(a+b)(a-b)=a^2-b^2\right]$

$=\frac{2\sqrt{10}-2\sqrt{6}}{5-3}$

$=\frac{2\left(\sqrt{10}-\sqrt{6}\right)}{2}$

$=\sqrt{10}-\sqrt{6}$

- **Laws of rational exponents of real numbers:**

  Let *a* and *b* be two real numbers and *m* and *n* be two rational numbers then

$a^p \cdot a^q = a^{p+q}$

$(a^p)^q = a^{pq}$

$\frac{a^p}{a^q} = a^{p-q}$

$a^p b^p = (ab)^p$

$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

$a^{-p} = \frac{1}{a^p}$

Example:

$\sqrt[3]{(512)^{-2}}$

$= \left[(512)^{-2}\right]^{\frac{1}{3}}$

$= (512)^{\frac{-2}{3}} \quad [(a^m)^n = a^{mn}]$

$= (8^3)^{\frac{-2}{3}}$

$= (8)^{3\times\frac{-2}{3}} \quad [(a^m)^n = a^{mn}]$

$= (8)^{-2}$

$= \frac{1}{8^2} \qquad \left[a^{-m} = \frac{1}{a^m}\right]$

$= \frac{1}{64}$